Towards a complete determination of the spectrum of a transfer operator associated with intermittency

2003 J. Phys. A: Math. Gen. 36 2455

(http://iopscience.iop.org/0305-4470/36/10/306)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 2455–2461 PII: S0305-4470(03)53616-2

Towards a complete determination of the spectrum of a transfer operator associated with intermittency

Thomas Prellberg

Institut für Theoretische Physik, Technische Universität Clausthal, Arnold-Sommerfeld Straße 6, D-38678 Clausthal-Zellerfeld, Germany

E-mail: thomas.prellberg@tu-clausthal.de

Received 13 September 2002, in final form 28 January 2003
Published 26 February 2003
Online at stacks.iop.org/JPhysA/36/2455

Abstract
The physical phenomenon of intermittency can be investigated via the spectral analysis of a transfer operator associated with the dynamics of an interval map with indifferent fixed point. For an example of such an intermittent map, the Farey map, we give a simple proof that the transfer operator is self-adjoint on a suitably defined Hilbert space and characterize its spectrum. Using a suitable first-return map, we present a highly efficient numerical method for the determination of all the eigenvalues, including those embedded in the continuous spectrum.

PACS numbers: 02.30.Sa, 02.30.Tb, 02.70.Hm, 05.10.−a, 05.45.Ac

1. Introduction

Intermittency, one of the main routes from order to chaos [1], is characterized by the loss of stability of a fixed point of the dynamics. The dynamics directly at the transition are determined by a marginally unstable fixed point near which trajectories are slowed down severely, leading to the intermittency characteristic interplay of chaotic and regular dynamics. The simplest type of intermittency, known as type I, is illustrated by a tangent bifurcation in a one-dimensional map [2]. Such a behaviour can easily be modelled by a map f of the unit interval [0, 1] which is uniformly expanding everywhere except near an indifferent fixed point at zero, where $f(x) \sim x + cx^{r+1}$ as $x \to 0$ with exponent $1 + r > 1$. A typical example is the Manneville map $f(x) = x + x^{1+r} \bmod 1$ [3].

There have been many theoretical approaches to the description of intermittency such as renormalization group analysis [4]. An approach suited to rigorous treatment is given by the thermodynamic formalism [5] and leads to the spectral analysis of transfer operators. In contrast with uniformly expanding maps, however, the indifferent fixed point induces non-Gibbsian equilibrium states or, more precisely, weakly Gibbsian states [6]. Therefore, deeper understanding based on rigorous analysis has been hard to come by. Some progress was made by studying a suitably defined piecewise linear interval map [7], albeit at the cost of simplifying the dynamics by severely reducing correlations. In [8, 9] we argued the disappearance of a spectral gap for the Perron–Frobenius operator, implying loss of an exponential decay of correlations for the dynamics. The decay of correlations has later been shown to follow a power law; in the analytic case ($r = 1$) a numerical estimate of t^{-2} [10] has been complemented by a rigorous upper bound of $t^{-2}\log t$, obtained by random perturbation techniques [11]. Only very recently has the Perron–Frobenius operator for a particular intermittent map, the Farey map, been shown to be self-adjoint on an appropriate Hilbert space [12] with continuous spectrum on the interval [0, 1]. There is also numerical work available describing exactly how the continuous spectrum emerges when approaching the intermittency transition [13].

Much less is known about the spectrum of the Ruelle–Perron–Frobenius (RPF) operator, which generalizes the Perron–Frobenius operator within the framework of the thermodynamic formalism. In [8, 9] we showed that for a general class of intermittent maps this operator is quasi-compact with essential spectral radius equal to 1, and that the leading eigenvalue undergoes a phase transition characteristic of intermittent dynamics. Only recently was it shown that for a class of piecewise analytic maps the continuous spectrum is restricted to the interval [0, 1] on an appropriate function space [14]. The present work gives for the first time a complete spectral analysis of the RPF operator for an intermittent map, the Farey map.

A promising conceptual approach to the study of intermittency is the study of a first-return map (or induced map) with respect to a domain of phase space away from the intermittent region. We accomplished this in [8, 9] by introducing a suitably modified transfer operator for the induced map and related its spectral properties to those of the transfer operator of the intermittent system. This approach has since been extended to the study of other quantities such as regularized Fredholm determinants and dynamical zeta functions [12, 15–17] and multi-dimensional systems [18]. One essential idea employed is that one can understand a general transfer operator $\mathcal{P}$ of an intermittent map by first considering its 'intermittent part' $\mathcal{P}_0$ and then viewing the 'chaotic remainder' $\mathcal{P}_1 = 1 - \mathcal{P}$ as a perturbation. In this way, many of the results presented here can, in principle, be extended to more general intermittent maps, although we shall focus our attention on the *Farey map* of the interval [0, 1] onto itself, which is defined as

$$f(x) = \begin{cases} f_0(x) = x/(1-x) & \text{if } 0 \leqslant x \leqslant \frac{1}{2} \\ f_1(x) = (1-x)/x & \text{if } \frac{1}{2} < x \leqslant 1. \end{cases} \tag{1}$$

We denote the inverses by $F_0(x) = f_0{}^{-1}(x) = x/(1+x)$ and $F_1(x) = f_1{}^{-1}(x) = 1/(1+x)$.

The advantage of this map is that each branch can be analytically extended to the whole Riemann sphere and that higher iterates of the left branch can be given exactly, $f_0{}^n(x) = x/(1-nx)$, a fact that is intimately connected to the global conjugacy of $f_0(x)$ to the shift $x \to x - 1$.[1] The thermodynamic formalism suggests the study of the RPF operator $\mathcal{P}$ associated with a map f, which is given by $\mathcal{P}\varphi(x) = \sum_{f(y)=x} |f'(y)|^{-\beta}\varphi(y)$ with $\beta \in \mathbb{R}$. For the Farey map, the operator consists of two terms that can be readily identified with the 'intermittent' and 'chaotic' parts. We thus write $\mathcal{P} = \mathcal{P}_0 + \mathcal{P}_1$ where $\mathcal{P}_i\varphi(x) = |F_i{}'(x)|^\beta \varphi(F_i(x))$ and $\beta \in \mathbb{R}$, i.e.

$$\mathcal{P}\varphi(x) = \frac{1}{(1+x)^{2\beta}}\left[\varphi\left(\frac{x}{1+x}\right) + \varphi\left(\frac{1}{1+x}\right)\right]. \tag{2}$$

Despite the apparent simplicity of this operator, a precise investigation of its properties has proved to be surprisingly difficult. In this paper, we present abstract arguments and numerical

[1] For general parabolic fixed points such a conjugacy can always be found locally, a fact that is used in the analysis of [14].

calculations to describe its spectral properties. In particular, we prove that $\mathcal{P}$ is self-adjoint on an explicitly given Hilbert space; on this Hilbert space, $\mathcal{P}$ has continuous spectrum on the unit interval. We then describe its spectral properties in more detail and proceed to determine the eigenvalues with a highly efficient numerical method based on the first-return map.

2. Spectral properties of the transfer operator

As indicated above, one can understand the spectral properties of $\mathcal{P}$ by first studying the operator $\mathcal{P}_0$ and then viewing $\mathcal{P}_1$ as a perturbation.

Under formal conjugacy with $\mathcal{C}\varphi(x) = \varphi(f_1(x))$, the operator $\mathcal{P}_0$ transforms to the shift operator $S\varphi(x) = \varphi(1 + x)$. This operator is now independent of β, and on a suitably defined function space has continuous spectrum $\sigma(S) = \sigma_c(S) = [0, 1]$. This can be understood by considering functions φ which are obtained by a generalized Laplace transform $\varphi(x) = \mathcal{L}\psi(x) = \int_0^\infty \mathrm{e}^{-sx}\psi(s)\,\mathrm{d}\mu(s)$ of square integrable functions $\psi \in L^2(\mathbb{R}_+, \mu)$. (This automatically ensures analyticity of $\varphi(x)$ in a suitable domain.) The action of the operator S is then conjugate to multiplication by e^{-s} on $L^2(\mathbb{R}_+, \mu)$. The spectrum of this multiplication operator is continuous and given by the closure of the range of the multiplying function. Using this line of reasoning, one can prove that $\mathcal{P}_0$ is a bounded self-adjoint operator on the Hilbert space $\mathcal{C}\mathcal{L}L^2(\mathbb{R}_+, \mu)$ with spectrum $\sigma(\mathcal{P}_0) = \sigma_c(\mathcal{P}_0) = [0, 1]$.

To study the spectrum of $\mathcal{P}$, we consider the identity

$$1 - z\mathcal{P} = (1 - z\mathcal{P}_0)(1 - \mathcal{M}_z) \tag{3}$$

with $\mathcal{M}_z = (1 - z\mathcal{P}_0)^{-1}z\mathcal{P}_1$ being an operator-valued analytic function for $z \in \mathbb{C} - [1, \infty)$.[2] Expanding formally in powers of z, we find that $\mathcal{M}_z = \sum_{n=1}^{\infty} z^n {\mathcal{P}_0}^{n-1}\mathcal{P}_1$ is a transfer operator associated with the induced map g on $\left[\frac{1}{2}, 1\right]$ with the branches $g_n = {f_0}^{n-1} f_1$ for $n \in \mathbb{N}$, where the return time n has been encoded via a multiplicative weight factor z^n.

To determine the spectral properties of the operator $\mathcal{M}_z$ we note that the induced map g on $\left[\frac{1}{2}, 1\right]$ is expanding. It follows that for $|z| < 1$ the operator $\mathcal{M}_z$ acting on a Frechet space of functions analytic in an open connected domain Ω containing the interval $\left[\frac{1}{2}, 1\right]$ is nuclear of order zero. Therefore, the analytic continuation of $\mathcal{M}_z$ is also nuclear of order zero (which can be seen via the analytic continuation of the associated Fredholm determinants, see [14]).

The identity (3) immediately implies that $\lambda = z^{-1}$ is an eigenvalue of $\mathcal{P}$ if and only if 1 is an eigenvalue of $\mathcal{M}_z$. The respective eigenspaces are identical, so that the geometric multiplicities of the respective eigenvalues z^{-1} and 1 are the same. Using the analytic continuation of $\mathcal{M}_z$, general analyticity arguments [19] and the nuclearity of $\mathcal{M}_z$ imply that the non-zero point spectrum of $\mathcal{P}$ consists of isolated eigenvalues with finite multiplicity, with 0 and 1 as the only possible accumulation points. (Nuclearity of $\mathcal{M}_z$ and analytic dependence of $\mathcal{M}_z$ on z imply that there are at most finitely many solutions to $1 \in \sigma(\mathcal{M}_z)$ in a small neighbourhood of z by standard perturbation theory results.) Moreover, bounds on $\mathcal{M}_z$ for $|z| < 1$ imply that there are only finitely many eigenvalues with modulus greater than 1. Along the cut $z \in (1, \infty)$ we see that if the analytic continuation of $\mathcal{M}_z$ does not have an eigenvalue 1 then $1 - \mathcal{M}_z$ is invertible, implying that $\sigma_c(\mathcal{P}) = \sigma_c(\mathcal{P}_0) = [0, 1]$, albeit with the possibility of embedded eigenvalues. Similar arguments can be found in detail for a rather general setting in [14]. We further know that zero is an eigenvalue of infinite multiplicity, as $\mathcal{P}\varphi = 0$ for any φ satisfying $\varphi(x) = -\varphi(1 - x)$.

We now show that $\mathcal{P}$ is in fact self-adjoint on a suitably defined Hilbert space for which we can give a representation of $\mathcal{P}$ which is explicitly symmetric. For this, we consider the transfer

[2] In [12] a related identity has been used, leading to the study of $\mathcal{M}'_z = \mathcal{P}_1(1 - z\mathcal{P}_0)^{-1}$. Obviously, $\mathcal{M}_z$ and $\mathcal{M}'_z$ are related by conjugacy.

operator Q for a general Möbius transformation $h(x) = (ax + b)/(cx + d)$ with $a, b, c, d \in \mathbb{R}$ and $\sigma = ad - bc = \pm 1$. We have $Q\varphi(x) = (a - cx)^{-2\beta}\varphi\left(\frac{dx-b}{a-cx}\right)$ and under a generalized Laplace transform,

$$\varphi(x) = \mathcal{L}\psi(x) = \int_0^\infty \mathrm{e}^{-sx}\psi(s)\,\mathrm{d}\mu(s) \tag{4}$$

with μ being the density of a measure on $\mathbb{R}_+$, Q is conjugated to an integral operator $\mathcal{K}\psi(s) = \int_0^\infty K(s, t)\psi(t)\,\mathrm{d}\mu(t)$ on $L^2(\mathbb{R}_+, \mu)$. Using the Schläfli integral representation for Bessel functions [20],

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{C}} \exp\left[\frac{x}{2}\left(z + \frac{\sigma}{z}\right)\right] z^{-\nu-1}\,\mathrm{d}z = \begin{cases} I_\nu(x) & \sigma = 1 \\ J_\nu(x) & \sigma = -1 \end{cases} \tag{5}$$

where $\mathcal{C}$ is a Hankel contour in the z-plane encircling counterclockwise a cut along the negative real axis; one can calculate for $c \neq 0$ directly

$$\mu(s)K(s, t) = \frac{1}{c}\exp\left[\frac{as + td}{c}\right]\left(\frac{s}{t}\right)^{\beta-\frac{1}{2}} Z_{2\beta-1}\left(\frac{2}{c}\sqrt{st}\right) \tag{6}$$

where $Z_\nu(u) = I_\nu(u)$ for $\sigma = 1$ and $Z_\nu(u) = J_\nu(u)$ for $\sigma = -1$, respectively. Demanding that the kernel be symmetric determines the appropriate measure. In this way, we obtain for $Q = \mathcal{P}_1$ the measure $\mu(s) = \mathrm{e}^{-s}s^{2\beta-1}$ and the kernel $K(s, t) = (st)^{\frac{1}{2}-\beta}J_{2\beta-1}(2\sqrt{st})$. Combining this with the discussion of the operator $\mathcal{P}_0$, we can define a function space on which $\mathcal{P}$ is self-adjoint by using the obvious self-adjointness of the bounded operator $\mathcal{S}$ acting on $L^2(\mathbb{R}_+, \mu)$, which is explicitly given as

$$\mathcal{S}\psi(s) = \mathrm{e}^{-s}\psi(s) + \int_0^\infty K(s, t)\psi(t)\,\mathrm{d}\mu(t) \tag{7}$$

with $\mu(s) = \mathrm{e}^{-s}s^{2\beta-1}$ and $K(s, t) = (st)^{\frac{1}{2}-\beta}J_{2\beta-1}(2\sqrt{st})$. It follows that $\mathcal{P}$ is self-adjoint on the Hilbert space

$$\left\{\varphi(x) = x^{-2\beta}\int_0^\infty \mathrm{e}^{-s\frac{1-x}{x}}\psi(s)\,\mathrm{d}\mu(s) : \psi \in L^2(\mathbb{R}_+, \mu)\right\} \tag{8}$$

with $\mu(s) = \mathrm{e}^{-s}s^{2\beta-1}$ and appropriate induced inner product. $\mathcal{M}_z$ also leaves this Hilbert space invariant, and via a similar argument one can show self-adjointness for real $z < 1$.

The case $\beta = 1$ has already been treated in [12], although with a slightly different choice of measure, motivated by the Hilbert space approach for the continued fraction transform [21, 22]. There it was shown that for $\tilde{\mu}(s) = s/(\mathrm{e}^s - 1)$ the only non-zero eigenvalue of $\mathcal{P}$ is 1 with eigenfunction $\varphi(x) = 1/x$. The measure $\mu(s) = s\,\mathrm{e}^{-s}$ considered here has the advantage that the self-adjointness is explicitly evident from (7). However, $\varphi(x) = 1/x$ is not an element of the Hilbert space considered here, as it corresponds to $\psi(s) = 1/s$ which has infinite norm in $L^2(\mathbb{R}_+, \mu)$. We find this quite natural, as the usual interpretation of this eigenfunction is as an absolutely continuous invariant density with respect to the Lebesgue measure on the interval [0, 1], which in this case is non-normalizable.

3. Numerical analysis of the spectrum

As shown above, $\mathcal{M}_z$ is an operator-valued analytic function with nuclear spectrum. By standard analytic perturbation theory [19], the eigenvalues of $\mathcal{M}_z$ are (branches of) analytic functions in z with only algebraic singularities. This therefore provides a possibility to compute the eigenvalues of $\mathcal{P}$ numerically; choosing an eigenvalue branch $\lambda_n(z)$, one simply needs to

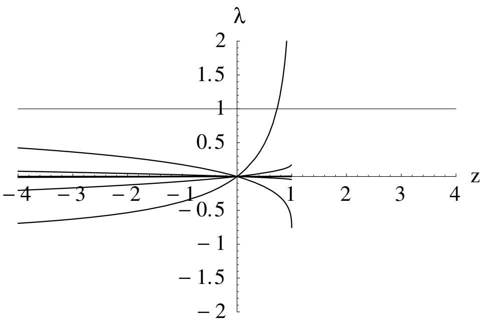

Figure 1. The seven leading eigenvalues of $\mathcal{M}_z$ for $\beta = 0.5$. The analytic continuation of the eigenvalue branches beyond $z = 1$ is complex valued and has a jump along the cut $[1, \infty)$. Only the largest eigenvalue branch intersects $\lambda = 1$.

solve $\lambda_n(z) = 1$ to obtain an eigenvalue $\lambda = 1/z$ of $\mathcal{P}$. The essential advantage over working directly with $\mathcal{P}$ is that we have removed the continuous spectrum which presents enormous difficulties for a direct numerical analysis.

As the operator $\mathcal{M}_z$ acts on a Hilbert space of analytic functions, it is reasonable to consider the action of $\mathcal{M}_z$ on coefficients of power series. From the explicit expansion $\mathcal{M}_z\varphi(x) = \sum_{n=1}^{\infty} z^n(1+nx)^{-2\beta}\varphi(1-x/(1+nx))$ we obtain by expanding $\varphi(x)$ in a power series around $x = 1$ matrix elements $\mathcal{M}_z^{n,m}$ in terms of the polylogarithm $\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$ as

$$M_z^{n,m} = \sum_{k=0}^{n} (-1)^{n-k} \binom{-2\beta - m}{k} \binom{-2\beta - k}{n-k} \left(\frac{1}{z}\mathrm{Li}_{2\beta+m+k}(z) - 1 \right). \qquad (9)$$

We then approximate $\mathcal{M}_z$ with truncated operators $\mathcal{M}_z^{(N)}$ acting on a subspace of polynomials of at most degree N. It turns out that this approximation works very well and that one obtains the values of the leading eigenvalues to a high accuracy. In this setting, this idea goes back to [15]. In [23], where a related operator was studied, the leading eigenvalues have been obtained in this way with accuracy of 10^{-25}.[3]

We begin the description of the results of our analysis by briefly considering the special cases $\beta = -N/2$ with $N \in \mathbb{N}_0$. For these values of β we find that polynomials of at most degree N give an $(N+1)$-dimensional invariant subspace for $\mathcal{M}_z$ and $\mathcal{P}$. The corresponding truncated matrix $\mathcal{M}_z^{(N)}$ has entries that are rational functions in z from which one can easily calculate $N+1$ eigenvalues exactly. Numerically, we observe that eigenfunctions of $\mathcal{M}_z$ which are not in this invariant subspace have eigenvalues that are strictly smaller in modulus. It is especially noteworthy that these leading eigenvalues can be analytically continued across $z = 1$, which allows for eigenvalues of the Farey operator $\mathcal{P}$ embedded in the continuous spectrum. However, a more detailed numerical analysis indicates that this analytic continuation is possible *only* for $\beta = -N/2$, and a small deviation from these values leads to eigenvalue branches $\lambda_n(z)$ whose analytic extension generically shows a non-vanishing jump in the imaginary part along the cut $[1, \infty)$. One can understand this heuristically by considering the analytic extension of $\mathrm{Li}_s(z)$, which along the cut $[1, \infty)$ jumps at $x > 1$ by an amount $2\pi\mathrm{i}\log^{s-1}(x)/\Gamma(s)$. For eigenvalues of $\mathcal{P}$ embedded in the continuous spectrum to exist, we need to solve $1 = \lambda_n(z)$, which is not possible if $\lambda_n(z)$ generically has a non-zero imaginary part. Therefore we conjecture that

[3] As in [23], we find that we can improve convergence by expanding around a different point. However, this leads to the introduction of spurious eigenvalues that make the analysis more difficult. By choosing an expansion around $x = 1$ we avoid spurious eigenvalues and find that, overall, the spectrum is better approximated.

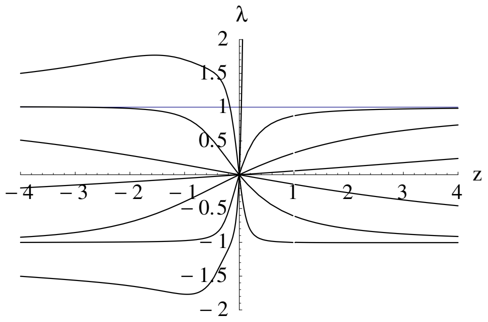

Figure 2. The seven leading eigenvalues of $\mathcal{M}_z$ for $\beta = -3$. Here, the eigenvalue branches can be continued beyond $z = 1$ and intersect $\lambda = 1$ at $z = -2.971$, $z = -0.168$, $z = 0.038$ and $z = 13.101$, the last value being outside the range of this plot.

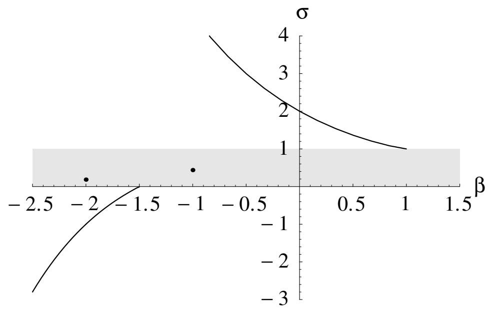

Figure 3. The spectrum of $\mathcal{P}$ as a function of β, showing two non-zero eigenvalue branches and isolated eigenvalues at $\beta = -1$ and $\beta = -2$ which are embedded in the continuous spectrum [0, 1].

$\lambda_n(z) = 1$ for real $z > 1$ can only be satisfied when $\beta = -N/2$, and that consequently no eigenvalues of $\mathcal{P}$ embedded in the continuous spectrum exist for other values of β. Thus, in what follows we shall mainly be concerned with real $z \leqslant 1$.

For $\beta > 1$ we find that all the eigenvalues of $\mathcal{M}_z$ are strictly less than 1 in modulus, so that $\mathcal{P}$ has continuous spectrum [0, 1] and no non-zero eigenvalues at all. As β decreases the eigenvalues of $\mathcal{M}_z$ increase in value. For $\beta > 1$, we find that the leading eigenvalue branch of $\mathcal{M}_z$ intersects $\lambda = 1$, leading to the emergence of a simple leading eigenvalue of $\mathcal{P}$ for $\beta < 1$.[4] The z-dependence of the spectrum of $\mathcal{M}_z$ is shown in figure 1 for $\beta = \frac{1}{2}$. Only the largest eigenvalue intersects $\lambda = 1$, implying that $\mathcal{P}$ has only one non-zero eigenvalue. At $\beta = -\frac{3}{2}$ a second eigenvalue branch begins to intersect $\lambda = 1$ at large negative z, implying that $\mathcal{P}$ has two non-zero eigenvalues. The second eigenvalue of $\mathcal{P}$ is negative and becomes in modulus equal to the essential spectral radius at $\beta = -2$, and therefore determines for $\beta < -2$ the spectral gap that controls the decay of correlations. Figure 2 illustrates the leading spectrum of $\mathcal{M}_z$ for $\beta = -3$. At this special value of β we can extend the eigenvalue branches beyond $z = 1$ and find a total of four eigenvalue branches that cross $\lambda = 1$. One of these crossings is at $z = 13.101$, corresponding to an eigenvalue of $\mathcal{P}$ embedded in the continuous spectrum.

[4] In [8, 9] it was proved that this leading eigenvalue decreases to 1 like $-(1-\beta)/\log(1-\beta)$ as β approaches 1 from below.

Proceeding in this fashion, we can numerically determine the complete spectrum of $\mathcal{P}$ for arbitrary real β. Figure 3 shows the spectrum obtained in this way for $-2.5 \leqslant \beta \leqslant 1.5$.

In summary, we have abstractly characterized the spectrum of the transfer operator for the Farey map and presented a highly efficient method for the explicit computation of its eigenvalues. Of special interest is the interplay of sub-dominant eigenvalues and the continuous spectrum. Both the rigorous arguments and numerical methods are in principle generalizable to more complicated systems with intermittency. However, a minimal requirement for the application of the methods presented here to prove self-adjointness properties is naturally that the branches of the map be real-analytic. If one considers changes in the class of maps, e.g. when considering a piecewise linear version of the Farey map, the whole structure of the sub-dominant spectrum changes [15]. This, however, is hardly surprising, as a linearization procedure suppresses correlations present in the original dynamics. In order to apply the numerical method to other systems, one needs good control of the transfer operator $\mathcal{M}_z$ for the induced system. This is the case if one has explicit formulae for higher iterates of the map, such as for the Farey map considered here.

References

[1] Schuster H G 1995 *Deterministic Chaos, An Introduction* (Weinheim: Physik Verlag)
[2] Poemau Y and Manneville P 1980 *Commun. Math. Phys.* **74** 189
[3] Manneville P 1980 *J. Physique* **41** 1235
[4] Hu B and Rudnick J 1982 *Phys. Rev. Lett.* **48** 1645
[5] Ruelle D 1978 *Thermodynamic Formalism* (Reading, MA: Addison-Wesley)
[6] Maes C, Redig F, Takens F, van Moffaert A and Verbitsky E 2000 *Nonlinearity* **13** 1681
[7] Gaspard P and Wang X J 1988 *Proc. Natl Acad. Sci. USA* **85** 4591
[8] Prellberg T 1991 *PhD Thesis* Virginia Tech
[9] Prellberg T and Slawny J 1992 *J. Stat. Phys.* **66** 503
[10] Lambert A, Siboni S and Vaienti S 1993 *J. Stat. Phys.* **72** 1305
[11] Liverani C, Saussol B and Vaienti S 1999 *Ergodic Theory Dyn. Syst.* **19** 671
[12] Isola S 2002 *Nonlinearity* **15** 1521
[13] Kaufmann Z, Lustfeld H and Bene J 1996 *Phys. Rev.* E **53** 1416
[14] Rugh H H 1999 *Invent. Math.* **135** 1
[15] Dodds P 1993 *Honour's Thesis* The University of Melbourne
[16] Dodds P 1993 *Master's Thesis* The University of Melbourne
[17] Isola S 1995 *Preprint* MPARC95265
[18] Pollicott M and Yuri M 2001 *Nonlinearity* **14** 1265
[19] Kato T 1980 *Perturbation Theory for Linear Operators* (Berlin: Springer)
[20] Arfken G B and Weber H J 2001 *Mathematical Methods for Physicists* (New York: Harcourt/Academic)
[21] Mayer D H and Roepstorff G 1987 *J. Stat. Phys.* **47** 149
[22] Mayer D H 1990 *Commun. Math. Phys.* **130** 311
[23] Daudé H, Flajolet P and Vallée B 1997 *Comb. Probab. Comput.* **6** 397